

# Sums of three quadratic endomorphisms of an infinite-dimensional vector space

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## Abstract

Let  $V$  be an infinite-dimensional vector space over a field. In a previous article [5], we have shown that every endomorphism of  $V$  splits into the sum of four square-zero ones but also into the sum of four idempotent ones. Here, we study decompositions into sums of three endomorphisms with prescribed split annihilating polynomials with degree 2. Except for endomorphisms that are the sum of a scalar multiple of the identity and of a finite-rank endomorphism, we achieve a simple characterization of such sums. In particular, we give a simple characterization of the endomorphisms that split into the sum of three square-zero ones, and we prove that every endomorphism of  $V$  is a linear combination of three idempotents.

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## 1 Introduction

Throughout the article,  $\mathbb{F}$  denotes an arbitrary field and  $V$  is an infinite-dimensional vector space over  $\mathbb{F}$ , whose algebra of endomorphisms we denote by  $\text{End}(V)$ . An endomorphism  $u$  of  $V$  is called **quadratic** whenever there exists a polynomial

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$p(t) \in \mathbb{F}[t]$  with degree 2 such that  $p(u) = 0$ . Special cases of quadratic endomorphisms are the square-zero ones, the idempotent ones, and the involutions. Let  $p_1, \dots, p_n$  be split polynomials with degree 2 over  $\mathbb{F}$ . We call an endomorphism  $u$  of  $V$  a  $(p_1, \dots, p_n)$ -**sum** whenever there exists an  $n$ -tuple  $(u_1, \dots, u_n)$  of endomorphisms of  $V$  such that

$$u = \sum_{k=1}^n u_k \quad \text{and} \quad \forall k \in \llbracket 1, n \rrbracket, p_k(u_k) = 0.$$

We adopt a similar definition for square matrices over  $\mathbb{F}$ .

Likewise, a scalar  $\lambda$  is called a  $(p_1, \dots, p_n)$ -**sum** whenever there exists an  $n$ -tuple  $(x_1, \dots, x_n) \in \mathbb{F}^n$  such that

$$\lambda = \sum_{k=1}^n x_k \quad \text{and} \quad \forall k \in \llbracket 1, n \rrbracket, p_k(x_k) = 0.$$

In a recent work [5], we have obtained the following general result:

**Theorem 1.** *Let  $(p_1, p_2, p_3, p_4)$  be a four-tuple of split polynomials with degree 2 over  $\mathbb{F}$ . Then, every endomorphism of an infinite-dimensional vector space over  $\mathbb{F}$  is a  $(p_1, p_2, p_3, p_4)$ -sum.*

In particular, every endomorphism of an infinite-dimensional vector space is the sum of four square-zero endomorphisms, but also of four idempotents, of two idempotents and two square-zero endomorphisms, etc. This contrasts with two results that were previously known:

- If  $V$  is a finite-dimensional vector space, then an endomorphism of  $V$  is the sum of four square-zero endomorphisms if and only if its trace equals zero (see [8] for the case of a complex vector space, and [3] for the general case).
- If  $\mathbb{F} = \mathbb{C}$  and  $V$  is a Hilbert space, then any bounded operator on  $V$  is the sum of five square-zero bounded operators [1], and a bounded operator on  $V$  is the sum of four square-zero bounded operators if and only if it is a commutator [8].

Compared to the latter result, Theorem 1 is purely algebraic, and no structure from analysis is involved.

In [5], it was shown through various examples that four summands are necessary in Theorem 1. To be more precise, if we have three split polynomials  $p_1, p_2, p_3$  with degree 2 over  $\mathbb{F}$ , in general there exist endomorphisms of  $V$  that fail to be  $(p_1, p_2, p_3)$ -sums. Thus, a natural question is whether a simple characterization of  $(p_1, p_2, p_3)$ -sums can be obtained. In this work, we shall obtain an answer that is very close to a positive one. More precisely, we shall obtain such a characterization if we exclude very specific endomorphisms, specifically those that split into  $\lambda \text{id}_V + w$  where  $\lambda$  is a scalar and  $w$  is a finite-rank endomorphism of  $V$ . For such special endomorphisms, no characterization appears possible in general, but in the special case when  $p_1 = p_2 = p_3 = t^2$  we shall nevertheless succeed in obtaining one, leading to a complete characterization of the sums of three square-zero endomorphisms. In addition, we will give a full characterization of the endomorphisms that split into the sum of three idempotents if the underlying field has characteristic 2, and we will prove that every endomorphism of an infinite-dimensional vector space is a linear combination of three idempotents, a result that was known to hold over finite-dimensional vector spaces [2, 4].

## 2 Main results, and the strategy

### 2.1 Main results

**Definition 1.** Let  $V$  be an infinite-dimensional vector space and  $u \in \text{End}(V)$ . A scalar  $\lambda$  is called a **dominant eigenvalue** of  $u$  if  $\text{rk}(u - \lambda \text{id}_V) < \dim V$ .

Note that this implies that  $\lambda$  is actually an eigenvalue of  $u$ , that is  $\text{Ker}(u - \lambda \text{id}_V) \neq \{0\}$ . Moreover,  $u$  has at most one dominant eigenvalue. Indeed, given distinct scalars  $\lambda$  and  $\mu$  we have  $\text{Ker}(u - \mu \text{id}_V) \subset \text{Im}(u - \lambda \text{id}_V)$ , and hence  $\text{rk}(u - \mu \text{id}_V) + \text{rk}(u - \lambda \text{id}_V) \geq \dim V$ : since  $V$  is infinite-dimensional, it follows that at most one of  $\text{rk}(u - \mu \text{id}_V)$  and  $\text{rk}(u - \lambda \text{id}_V)$  is less than  $\dim V$ .

Here is our first major result:

**Theorem 2.** *Let  $u$  be an endomorphism of an infinite-dimensional vector space over  $\mathbb{F}$ , with no dominant eigenvalue. Let  $p_1, p_2, p_3$  be split polynomials of degree 2 over  $\mathbb{F}$ . Then,  $u$  is a  $(p_1, p_2, p_3)$ -sum.*

Knowing this result, it only remains to understand when an endomorphism with a dominant eigenvalue is a  $(p_1, p_2, p_3)$ -sum. To do this, we recall that the

**trace** of a monic polynomial  $p \in \mathbb{F}[t]$  with degree  $n > 0$  is defined as the opposite of the coefficient of  $p$  on  $t^{n-1}$ . The trace of a nonconstant polynomial  $p$  with leading coefficient  $\alpha$  is defined as the one of  $\alpha^{-1}p$ , and denoted by  $\text{tr } p$ .

When we have an endomorphism  $u$  with a dominant eigenvalue  $\lambda$ , the following result gives a necessary condition on  $\lambda$  for  $u$  to be a  $(p_1, p_2, p_3)$ -sum.

**Theorem 3.** *Let  $V$  be an infinite-dimensional vector space over  $\mathbb{F}$ . Let  $p_1, p_2, p_3$  be split polynomials with degree 2 over  $\mathbb{F}$ . Let  $u$  be an endomorphism of  $V$  with a dominant eigenvalue  $\lambda$ , and assume that  $u$  is a  $(p_1, p_2, p_3)$ -sum. Then,  $\lambda$  is a  $(p_1, p_2, p_3)$ -sum or  $2\lambda = \text{tr } p_1 + \text{tr } p_2 + \text{tr } p_3$ .*

The above condition turns out to be sufficient unless  $u - \lambda \text{id}_V$  has finite rank:

**Theorem 4.** *Let  $V$  be an infinite-dimensional vector space over  $\mathbb{F}$ . Let  $p_1, p_2, p_3$  be split polynomials with degree 2 over  $\mathbb{F}$ . Let  $u$  be an endomorphism of  $V$  with a dominant eigenvalue  $\lambda$  such that  $u - \lambda \text{id}_V$  has infinite rank. Assume that  $\lambda$  is a  $(p_1, p_2, p_3)$ -sum or  $2\lambda = \text{tr } p_1 + \text{tr } p_2 + \text{tr } p_3$ . Then,  $u$  is a  $(p_1, p_2, p_3)$ -sum.*

Hence, it only remains to understand when the sum of  $\lambda \text{id}_V$  with a finite-rank endomorphism is a  $(p_1, p_2, p_3)$ -sum. We will show in Section 4 that this amounts to determine, given a scalar  $\lambda$  that satisfies the condition from Theorem 4, for which square matrices  $A \in M_n(\mathbb{F})$  there exists an integer  $q \geq 0$  such that  $(A + \lambda I_n) \oplus \lambda I_q$  is a  $(p_1, p_2, p_3)$ -sum, a problem that is open for general values of  $(p_1, p_2, p_3)$ . Nevertheless, for very specific values of  $(p_1, p_2, p_3)$  we shall obtain a complete characterization. Our first complete result deals with the case when  $p_1 = p_2 = p_3 = t^2$ , i.e. we will completely characterize the endomorphisms that split into the sum of three square-zero endomorphisms:

**Theorem 5.** *Let  $u \in \text{End}(V)$ , where  $V$  is an infinite-dimensional vector space over  $\mathbb{F}$ . Then,  $u$  is the sum of three square-zero endomorphisms if and only if none of the following situations occurs:*

- (i) *There exists  $\lambda \in \mathbb{F}$  together with a finite-rank endomorphism  $w \in \text{End}(V)$  such that  $\text{tr } w \notin \{0, \lambda\}$  and  $u = \lambda \text{id}_V + w$ .*
- (ii) *The characteristic of  $\mathbb{F}$  differs from 2 and  $u$  has a non-zero dominant eigenvalue.*

Below, we rewrite this result by discussing whether the ground field has characteristic 2 or not.

**Corollary 6.** *Let  $V$  be an infinite-dimensional vector space over a field  $\mathbb{F}$  with characteristic different from 2, and let  $u \in \text{End}(V)$ . Then,  $u$  is the sum of three square-zero endomorphisms if and only if it satisfies none of the following conditions:*

- (i)  $u$  has finite rank and non-zero trace;
- (ii)  $u$  has a non-zero dominant eigenvalue.

**Corollary 7.** *Let  $V$  be an infinite-dimensional vector space over a field  $\mathbb{F}$  with characteristic 2, and let  $u \in \text{End}(V)$ . Then,  $u$  is the sum of three square-zero endomorphisms if and only if there is no scalar  $\lambda$  such that  $u - \lambda \text{id}_V$  has finite rank and trace different from 0 and  $\lambda$ .*

Our second special case is the one when  $p_1 = p_2 = p_3 = t^2 - t$ , over fields with characteristic 2.

**Theorem 8.** *Let  $V$  be an infinite-dimensional vector space over a field  $\mathbb{F}$  with characteristic 2, and let  $u \in \text{End}(V)$ . Then,  $u$  is the sum of three idempotent endomorphisms of  $V$  if and only if none of the following conditions holds:*

- (i)  $u$  has a dominant eigenvalue outside of  $\{0_{\mathbb{F}}, 1_{\mathbb{F}}\}$ ;
- (ii) There exists  $\lambda \in \{0_{\mathbb{F}}, 1_{\mathbb{F}}\}$  such that  $u - \lambda \text{id}_V$  has finite rank and trace outside of  $\{0_{\mathbb{F}}, 1_{\mathbb{F}}\}$ .

Our final result generalizes one that was known over finite-dimensional spaces [2, 4]:

**Theorem 9.** *Let  $V$  be a vector space over a field. Then, every endomorphism of  $V$  is a linear combination of three idempotent endomorphisms.*

## 2.2 Some basic remarks and notation

Following the French convention, we denote by  $\mathbb{N}$  the set of all non-negative integers, by  $\mathbb{N}^*$  the set of all positive ones, and we use the word “countable” to mean “infinite countable”, and “uncountable” to mean “infinite uncountable”. Throughout the article,  $t$  denotes an indeterminate,  $\mathbb{F}[t]$  the algebra of polynomials in the indeterminate  $t$  and, given a non-negative integer  $d$ , we denote by  $\mathbb{F}_d[t]$  the linear subspace of  $\mathbb{F}[t]$  consisting of all polynomials with degree at most  $d$ .

Throughout the article, we will make frequent use of the following basic remarks.

*Remark 1.* Let  $u$  be an endomorphism of a vector space  $V$ . Let  $p_1, \dots, p_r$  be polynomials in  $\mathbb{F}[t]$ . Assume that  $V$  splits into  $V = \bigoplus_{i \in I} V_i$  in which each  $V_i$  is stable under  $u$  and the resulting endomorphism is denoted by  $u_i$ . Assume that, for all  $i \in I$ , the endomorphism  $u_i$  splits into  $u_i = \sum_{k=1}^r u_{i,k}$ , where  $u_{i,k} \in \text{End}(V)$  and  $p_k(u_{i,k}) = 0$  for all  $k \in \llbracket 1, r \rrbracket$ . Then, by setting  $u^{(k)} := \bigoplus_{i \in I} u_{i,k}$  for all  $k \in \llbracket 1, r \rrbracket$ , we see that  $u = \sum_{k=1}^r u^{(k)}$  and  $p_k(u^{(k)}) = 0$  for all  $k \in \llbracket 1, r \rrbracket$ .

*Remark 2* (Reduction to the monic case). Let  $p$  be a non-zero polynomial, with leading coefficient  $\lambda$ . An endomorphism is annihilated by  $p$  if and only if it is annihilated by  $\lambda^{-1}p$ , a polynomial which has the same degree and trace as  $p$ , and is split if and only if  $p$  is split. Hence, in the proof of all the above theorems, it will suffice to consider the case when the polynomials under consideration are all monic.

*Remark 3* (The canonical situation). In Theorem 2, no generality is lost in assuming that each polynomial under consideration is of the form  $t^2 - at$  for some  $a \in \mathbb{F}$ . Indeed, let  $p_1, p_2, p_3$  be split polynomials with degree 2 over  $\mathbb{F}$ . Given  $k \in \{1, 2, 3\}$ , we denote the roots of  $p_k$  by  $x_k, y_k$ . An endomorphism  $v$  is annihilated by  $p_k$  if and only if  $q_k := t^2 - (y_k - x_k)t$  annihilates  $v - x_k \text{id}$ . It follows that an endomorphism  $u \in \text{End}(V)$  is a  $(p_1, p_2, p_3)$ -sum if and only if  $u - (x_1 + x_2 + x_3) \text{id}_V$  is a  $(q_1, q_2, q_3)$ -sum. Moreover, it is obvious that  $u - (x_1 + x_2 + x_3) \text{id}_V$  has a dominant eigenvalue if and only if  $u$  does have one. In particular, in Theorem 2 it will suffice to consider the situation where each  $p_i$  equals  $t^2 - a_i t$  for some scalar  $a_i$ .

### 2.3 Strategy, and structure of the article

Our main strategy for the proof of Theorem 2 is globally similar to the one that was used in [5]. First of all, let  $u$  be an endomorphism of a vector space  $V$  over  $\mathbb{F}$ . The vector space structure of  $V$  is enriched into an  $\mathbb{F}[t]$ -module  $V^u$  by setting  $tx := u(x)$  for all  $x \in V$ . We say that  $u$  is **elementary** when  $V^u$  is a free  $\mathbb{F}[t]$ -module. A basic result that was proved in [5] (Theorem 1 in that article) reads as follows:

**Theorem 10.** *Let  $u$  be an elementary endomorphism of a vector space  $V$ , and  $p_1, p_2$  be split polynomials with degree 2 over  $\mathbb{F}$ . Then,  $u$  is a  $(p_1, p_2)$ -sum.*

In [5], the basic strategy then consisted in showing that, given split polynomials  $p_3, p_4$  with degree 2 over  $\mathbb{F}$  and an endomorphism  $u$  of  $V$ , there exist endomorphisms  $u_3$  and  $u_4$  of  $V$  such that  $u - u_3 - u_4$  is elementary and  $p_3(u_3) = p_4(u_4) = 0$  (note that in [7], Shitov proved the more powerful result that  $u$  is actually the sum of two elementary endomorphisms). Here, the strategy will be to start from an endomorphism  $u$  of  $V$  and from a split polynomial  $p$  with degree 2 over  $\mathbb{F}$ , and to search for an endomorphism  $v$  of  $V$  such that  $u - v$  is elementary and  $p(v) = 0$ . A definition is relevant here:

**Definition 2.** Let  $V$  be an  $\mathbb{F}$ -vector space,  $u$  be an endomorphism of  $V$ , and  $a$  be a scalar. We say that  $u$  is  **$a$ -elementarily decomposable** whenever there exists an endomorphism  $v$  of  $V$  such that  $v^2 = av$  and  $u - v$  is elementary.

Combining Theorem 10 with Remark 3, one sees that, in order to prove Theorem 2, it only remains to establish the following result:

**Theorem 11.** *Let  $u$  be an endomorphism of an infinite-dimensional vector space  $V$ , with no dominant eigenvalue. Then, for any scalar  $a$ , the endomorphism  $u$  is  $a$ -elementarily decomposable.*

In order to find a well-suited  $v$ , we shall use similar methods as in [5]: they involve *stratifications* of the  $\mathbb{F}[t]$ -module  $V^u$  and *connectors*. We will review the relevant definitions and results on them in Section 6, and we will also give new results that are needed here. The key notion is the one of a *good stratification*, that is defined in Section 6.2.

The article is laid out as follows. Section 3 essentially consists of a proof of Theorem 3 but also includes a technical lemma (the invariant subspace lemma) that will be used in later sections, together with a characterization of the scalar multiples of the identity that are  $(p_1, p_2, p_3)$ -sums. In Section 4, we will discuss what should be done in general to tackle the case of the sum of a scalar multiple of the identity with a finite-rank endomorphism. In Section 5, we will derive Theorem 4 from Theorem 2 and from the characterization of the  $(p_1, p_2, p_3)$ -sums among the scalar multiples of the identity.

The rest of the article is mainly devoted to the proof of Theorem 11. Section 6 consists of a discussion of stratifications and connectors. In Section 7, we will prove Theorem 11 in the special case of a vector space with uncountable dimension. Section 8 deals with the difficult case of a vector space with countable dimension. In the final section (Section 9), we shall complete the study by proving Theorems 5, 8 and 9, with the help of recent results on the finite-dimensional case [3].

### 3 Necessary conditions

#### 3.1 The case of scalar multiples of the identity

**Proposition 12.** *Let  $V$  be an infinite-dimensional vector space over  $\mathbb{F}$ , and  $\lambda \in \mathbb{F}$ . Let  $p_1, p_2, p_3$  be split polynomials with degree 2 over  $\mathbb{F}$ . Then,  $\lambda \text{id}_V$  is a  $(p_1, p_2, p_3)$ -sum if and only if  $\lambda$  is a  $(p_1, p_2, p_3)$ -sum or  $2\lambda = \text{tr } p_1 + \text{tr } p_2 + \text{tr } p_3$ .*

*Proof.* By Remark 2, we lose no generality in assuming that  $p_1, p_2, p_3$  are all monic, and we shall denote their respective traces by  $\alpha, \beta, \gamma$ .

We start with the converse implication. If  $\lambda$  is a  $(p_1, p_2, p_3)$ -sum, we split it up into  $\lambda = x_1 + x_2 + x_3$  where  $p_i(x_i) = 0$  for all  $i \in \llbracket 1, 3 \rrbracket$  and it is then obvious from writing  $\lambda \text{id}_V = x_1 \text{id}_V + x_2 \text{id}_V + x_3 \text{id}_V$  that  $\lambda \text{id}_V$  is a  $(p_1, p_2, p_3)$ -sum. Assume now that  $2\lambda = \alpha + \beta + \gamma$ . Write  $p_1(t) = (t-x)(t-y)$  with  $(x, y) \in \mathbb{F}^2$ . It is obvious that we can find scalars  $\mu$  and  $\nu$  such that the matrices  $B := \begin{bmatrix} 0 & \mu \\ 1 & \beta \end{bmatrix}$  and  $C := \begin{bmatrix} \lambda - x & \nu \\ -1 & \gamma + x - \lambda \end{bmatrix}$  are respectively annihilated by  $p_2$  and  $p_3$  (one simply chooses  $\mu$  and  $\nu$  such that the determinants of those matrices are, respectively,  $p_2(0)$  and  $p_3(0)$ , and one concludes thanks to the Cayley-Hamilton theorem). Then, since  $2\lambda = x + y + \beta + \gamma$ , we have

$$\lambda I_2 - B - C = \begin{bmatrix} x & -\mu - \nu \\ 0 & y \end{bmatrix}.$$

Once more, the Cayley-Hamilton theorem yields that  $A := \lambda I_2 - B - C$  is annihilated by  $p_1$ . Hence,  $\lambda I_2$  is a  $(p_1, p_2, p_3)$ -sum. It follows that, for every 2-dimensional vector space  $P$ , the endomorphism  $\lambda \text{id}_P$  is a  $(p_1, p_2, p_3)$ -sum.

Now, since  $V$  is infinite-dimensional we can split it up as  $V = \bigoplus_{i \in I} P_i$  where each  $P_i$  is a 2-dimensional vector space. Then,  $\lambda \text{id}_{P_i}$  is a  $(p_1, p_2, p_3)$ -sum for all  $i \in I$ , and by Remark 1 we conclude that  $\lambda \text{id}_V$  is a  $(p_1, p_2, p_3)$ -sum.

We now turn to the direct implication. Assume that  $\lambda \text{id}_V = a + b + c$  for some triple  $(a, b, c) \in \text{End}(V)^3$  such that  $p_1(a) = 0$ ,  $p_2(b) = 0$  and  $p_3(c) = 0$ . Assume also that  $2\lambda \neq \alpha + \beta + \gamma$ . Then, we shall prove that  $\lambda$  is a  $(p_1, p_2, p_3)$ -sum. Note first that  $b$  and  $c$  commute with  $u := (b+c)((\beta+\gamma) \text{id}_V - b - c)$  (see Lemma 3 of [6]). Indeed, by expanding we get that  $u = \gamma b + \beta c - bc - cb + \delta \text{id}_V$  for some  $\delta \in \mathbb{F}$ , and one checks that  $b$  commutes with  $\beta c - bc - cb$ : indeed,  $b(\beta c - bc - cb) = \beta bc - b^2 c - bcb = p_2(0)c - bcb$  and likewise  $(\beta c - bc - cb)b =$



$p_2(0)c - bcb$ ; it follows that  $b$  commutes with  $u$ , and likewise  $c$  commutes with  $u$ .

Next, we use  $b + c = \lambda \text{id}_V - a$  to obtain

$$u = (\lambda \text{id}_V - a)((\beta + \gamma - \lambda) \text{id}_V + a).$$

By expanding and using  $a^2 \in \alpha a + \mathbb{F} \text{id}_V$ , we get that

$$u = (2\lambda - \alpha - \beta - \gamma) a + \delta' \text{id}_V$$

for some  $\delta' \in \mathbb{F}$ . Since  $2\lambda - \alpha - \beta - \gamma \neq 0$ , we deduce that  $b$  and  $c$  both commute with  $a$ .

Symmetrically, we obtain that  $b$  commutes with  $c$ . Now, classically since  $a, b, c$  are pairwise commuting endomorphisms of a non-zero vector space that are annihilated by split polynomials, they have a common eigenvector; denoting by  $x, y, z$  the corresponding eigenvalue for  $a, b, c$ , respectively, we deduce that  $\lambda = x + y + z$ , which shows that  $\lambda$  is a  $(p_1, p_2, p_3)$ -sum.  $\square$

### 3.2 The invariant subspace lemma

**Lemma 13** (Invariant subspace lemma). *Let  $V$  be an infinite-dimensional vector space over  $\mathbb{F}$ . Let  $a, b, c$  be quadratic endomorphisms of  $V$ . Let  $\lambda \in \mathbb{F}$  and  $w \in \text{End}(V)$  be such that:*

$$(i) \quad \lambda \text{id}_V + w = a + b + c;$$

$$(ii) \quad \text{rk } w < \dim V.$$

*Let  $W$  be a linear subspace of  $V$  that includes  $\text{Im } w$  and such that  $\dim W < \dim V$ . Then, there exists a linear subspace  $\overline{W}$  of  $V$  such that  $\dim \overline{W} < \dim V$ ,  $\overline{W}$  includes  $W$  and is stable under  $a, b$  and  $c$ . Moreover if  $W$  is finite-dimensional then  $\overline{W}$  can be chosen finite-dimensional.*

*Proof.* We set

$$\overline{W} := W + a(W) + b(W) + c(W) + \sum_{(e,f) \in \{a,b,c\}^2} (ef)(W).$$

Each vector space in this sum has dimension less than or equal to  $\dim W$ . Hence,  $\dim \overline{W} < \dim V$  because  $V$  is infinite-dimensional. Moreover if  $W$  is finite-dimensional then so is  $\overline{W}$ .

Since  $\overline{W}$  includes  $W$ , it only remains to show that  $\overline{W}$  is stable under  $a, b, c$ . Obviously, it suffices to show that  $(efg)(W) \subset \overline{W}$  for all  $e, f, g$  in  $\{a, b, c\}$ . Note first that if  $e = f$ , then  $efg \in \text{span}(eg, g)$  since  $e$  is quadratic, whence  $(efg)(W) \subset \overline{W}$ . Likewise, this inclusion also holds if  $f = g$ .

Next,

$$\begin{aligned} ab + ba &= (a + b)^2 - a^2 - b^2 = (\lambda \text{id}_V + w - c)^2 - a^2 - b^2 \\ &= \lambda^2 \text{id}_V + 2\lambda w + w^2 - 2\lambda c - wc - cw + c^2 - a^2 - b^2. \end{aligned}$$

Since  $a, b, c$  are all quadratic and  $W$  includes  $\text{Im } w$ , it follows that

$$(ab + ba)(W) \subset W + a(W) + b(W) + c(W).$$

Next,  $aba = a(ab + ba) - a^2b \in \text{span}(a(ab + ba), ab, b)$ , and we deduce from this equality and from the previous inclusion that

$$(aba)(W) \subset a(W) + a^2(W) + (ab)(W) + (ac)(W) + (ab)(W) + b(W).$$

Since  $a^2$  is quadratic,  $a^2(W) \subset a(W) + W$ , and hence

$$(aba)(W) \subset \overline{W}.$$

More generally, we obtain  $(efe)(W) \subset \overline{W}$  for all  $(e, f) \in \{a, b, c\}^2$  (the case when  $e = f$  has already been dealt with). Finally,

$$\begin{aligned} cba &= (\lambda \text{id}_V + w - a - b)(ba) = \lambda(ba) + w(ba) - (aba) - b^2a \\ &\in \text{span}(ba, w(ba), aba, b^2a). \end{aligned}$$

We have already seen that the image of  $W$  under each endomorphism  $ba, w(ba), aba, b^2a$  is included in  $\overline{W}$ , whence  $(cba)(W) \subset \overline{W}$ . Hence, symmetrically  $(efg)(W) \subset \overline{W}$  for all distinct  $e, f, g$  in  $\{a, b, c\}$ . We conclude that  $\overline{W}$  is stable under  $a, b$  and  $c$ , which completes the proof.  $\square$

### 3.3 Proof of Theorem 3

Here, we derive Theorem 3 from the preceding two results. Let  $p_1, p_2, p_3$  be split polynomials of  $\mathbb{F}[t]$  with degree 2, and let  $u$  be an endomorphism of an infinite-dimensional vector space  $V$ . Assume that  $u$  has a dominant eigenvalue  $\lambda$  and that there exist endomorphisms  $a, b, c$  of  $V$  such that  $u = a + b + c$  and  $p_1(a) = p_2(b) = p_3(c) = 0$ .

Set  $w := u - \lambda \text{id}_V$ . By Lemma 13 applied to  $W := \text{Im } w$ , there exists a linear subspace  $\overline{W}$  of  $V$  that includes  $\text{Im } w$ , is stable under  $a, b$  and  $c$ , and whose dimension is less than the one of  $V$ . It follows that  $a, b, c$  induce endomorphisms of the infinite-dimensional quotient space  $\overline{V} := V/\overline{W}$  whose sum equals  $\lambda \text{id}_{\overline{V}}$ . By Proposition 12, we deduce that  $\lambda$  is a  $(p_1, p_2, p_3)$ -sum or  $2\lambda = \text{tr } p_1 + \text{tr } p_2 + \text{tr } p_3$ . This completes the proof of Theorem 3.

## 4 The case of the sum of a scalar multiple of the identity with a finite-rank endomorphism

In this section, we shall give a partial result to the problem of determining when an endomorphism of the form  $\lambda \text{id}_V + w$ , where  $\lambda$  is a scalar and  $w$  is a finite-rank endomorphism, is a  $(p_1, p_2, p_3)$ -sum.

The following definition is relevant to this problem:

**Definition 3.** Let  $A$  be an  $n$ -by- $n$  matrix with entries in  $\mathbb{F}$ , and let  $\lambda \in \mathbb{F}$ . Let  $p_1, p_2, p_3$  be split polynomials with degree 2 over  $\mathbb{F}$ . Let  $\lambda \in \mathbb{F}$  be such that  $2\lambda = \text{tr } p_1 + \text{tr } p_2 + \text{tr } p_3$  or  $\lambda$  is a  $(p_1, p_2, p_3)$ -sum. We say that  $A$  is a  **$(p_1, p_2, p_3)$ -sum  $\lambda$ -stably** if there exists a non-negative integer  $q$  such that the block-diagonal matrix  $(A + \lambda I_n) \oplus \lambda I_q$  is a  $(p_1, p_2, p_3)$ -sum.

Next, to any finite-rank endomorphism  $w$  of  $V$  can be attached a similarity class of square matrices as follows: we choose a minimal (finite-dimensional) linear subspace  $W$  of  $V$  such that  $\text{Im } w \subset W$  and  $W + \text{Ker } w = V$ . The dimension of  $W$  does not depend on the specific choice of  $W$  and we denote it by  $n(w)$ . Then, the similarity class of matrices that is attached to the induced endomorphism  $w|_W$  does not depend on the choice of  $W$  either. We denote this similarity class by  $[w]$ . Moreover, if  $W'$  is an arbitrary finite-dimensional linear subspace of  $V$  such that  $\text{Im } w \subset W'$  and  $W' + \text{Ker } w = V$ , any matrix that represents  $w|_{W'}$  has the form  $M \oplus 0_q$  for some  $M \in [w]$  and some non-negative integer  $q$ .

Here is our partial result:

**Theorem 14.** *Let  $V$  be an infinite-dimensional vector space,  $w$  be a finite-rank endomorphism of  $V$  and  $\lambda$  be a scalar. Let  $p_1, p_2, p_3$  be split polynomials with degree 2 over  $\mathbb{F}$ . Choose a matrix  $A$  in  $[w]$ . Then, the following conditions are equivalent:*

- (i) *The endomorphism  $\lambda \text{id}_V + w$  is a  $(p_1, p_2, p_3)$ -sum.*

(ii) The matrix  $A$  is a  $(p_1, p_2, p_3)$ -sum  $\lambda$ -stably, and either  $\lambda$  is a  $(p_1, p_2, p_3)$ -sum or  $2\lambda = \text{tr } p_1 + \text{tr } p_2 + \text{tr } p_3$ .

*Proof.* Set  $u := \lambda \text{id}_V + w$ .

Assume first that condition (ii) holds. Choose a non-negative integer  $q$  such that  $(A + \lambda I_{n(w)}) \oplus \lambda I_q$  is a  $(p_1, p_2, p_3)$ -sum. We have a finite-dimensional linear subspace  $W$  of  $V$  such that  $\text{Im } w \subset W$ ,  $W + \text{Ker } w = V$ , and  $A$  represents the endomorphism of  $W$  induced by  $w$ . Then, we can split  $V = W \oplus W'$  where  $W' \subset \text{Ker } w$ . We can further split  $W' = W'_1 \oplus W'_2$  so that  $W'_1$  has dimension  $q$ . Hence,  $V = W \oplus W'_1 \oplus W'_2$  and  $W'_2$  is infinite-dimensional. Since  $(A \oplus \lambda I_{n(w)}) \oplus \lambda I_q$  is a  $(p_1, p_2, p_3)$ -sum, the endomorphism  $u|_{W \oplus W'_1}$  is a  $(p_1, p_2, p_3)$ -sum. Moreover, by Proposition 12, the endomorphism  $\lambda \text{id}_{W'_2}$  of  $W'_2$  is a  $(p_1, p_2, p_3)$ -sum. Since  $u|_{W'_2} = \lambda \text{id}_{W'_2}$ , we deduce from Remark 1 that  $u$  is a  $(p_1, p_2, p_3)$ -sum.

Conversely, assume that condition (i) holds. By Theorem 3, we already know that  $\lambda$  is a  $(p_1, p_2, p_3)$ -sum or  $2\lambda = \text{tr } p_1 + \text{tr } p_2 + \text{tr } p_3$ .

Let  $a, b, c$  be endomorphisms of  $V$  such that  $u = a + b + c$  and  $p_1(a) = p_2(b) = p_3(c) = 0$ . Choosing a complementary subspace  $G$  of  $\text{Ker } w$  in  $V$  and applying the invariant subspace lemma to the finite-dimensional subspace  $W := \text{Im } w + G$ , we obtain a finite-dimensional linear subspace  $W'$  of  $V$  that is stable under  $a, b, c$ , includes  $\text{Im } w$  and satisfies  $W' + \text{Ker } w = V$ . Then, we can split  $V = W' \oplus V'$  such that  $V' \subset \text{Ker } w$ . Choosing a matrix  $A$  in  $[w]$ , it follows that, for some non-negative integer  $q$ , the matrix  $(A + \lambda I_{n(w)}) \oplus (\lambda I_q)$  represents the endomorphism of  $W'$  induced by  $u$ . Since  $a, b, c$  stabilize  $W'$ , this endomorphism turns out to be a  $(p_1, p_2, p_3)$ -sum, whence  $(A + \lambda I_{n(w)}) \oplus (\lambda I_q)$  is a  $(p_1, p_2, p_3)$ -sum, and we conclude that  $A$  is a  $(p_1, p_2, p_3)$ -sum  $\lambda$ -stably.  $\square$

Hence, in order to detect the  $(p_1, p_2, p_3)$ -sums among the endomorphisms of type  $\lambda \text{id}_V + w$ , with  $w$  of finite rank, it remains to understand which square matrices over  $\mathbb{F}$  are  $(p_1, p_2, p_3)$ -sums  $\lambda$ -stably. For general values of  $p_1, p_2, p_3$ , the latter problem is open, and probably intractable. For specific values of  $(p_1, p_2, p_3)$ , the recent [3] provides some answers.

## 5 Deriving Theorem 4 from Theorem 2

**Lemma 15** (The reduction lemma). *Let  $u \in \text{End}(V)$ , where  $V$  is an infinite-dimensional vector space over  $\mathbb{F}$ . Assume that  $u$  has a dominant eigenvalue  $\lambda$  and that  $u - \lambda \text{id}_V$  has infinite rank. Then, there exists a decomposition  $V = V_1 \oplus V_2$  into linear subspaces that are stable under  $u$  and such that:*

- (i)  $V_1$  is infinite-dimensional.
- (ii)  $u|_{V_1}$  has no dominant eigenvalue.
- (iii)  $u(x) = \lambda x$  for all  $x \in V_2$ .

*Proof.* Replacing  $u$  with  $u - \lambda \text{id}_V$ , no generality is lost in assuming that  $\lambda = 0$ . Denote by  $\text{rk } u$  the rank of  $u$ .

We choose a complementary subspace  $W$  of  $\text{Ker}(u)$  in  $V$ . By the rank theorem,  $W$  has dimension  $\text{rk } u$ , which is less than  $\dim V$  and hence than  $\dim \text{Ker}(u)$ . Hence, we can also choose a linear subspace  $W'$  of  $\text{Ker } u$  such that  $\dim W' = \text{rk } u$  and  $W' \cap (W + \text{Im } u) = \{0\}$ . Set  $V_1 := W' \oplus (W + \text{Im}(u))$ . The linear subspace  $V_1$  is stable under  $u$  since it includes  $\text{Im } u$ . As  $V_1 + \text{Ker}(u) = V$ , we can choose a linear subspace  $V_2$  of  $\text{Ker } u$  such that  $V_1 \oplus V_2 = V$ . Hence,  $V_2$  is stable under  $u$  and  $u(x) = \lambda x$  for all  $x \in V_2$ . It remains to show that  $V_1$  has the claimed properties.

Denote by  $u_1$  the endomorphism of  $V_1$  induced by  $u$ . Remember that  $u$  has infinite rank. Note that  $\dim V_1 = \dim W' = \text{rk } u$  and that  $\text{Im } u = \text{Im } u_1$  since  $u$  vanishes everywhere on  $V_2$ . Hence, 0 is not a dominant eigenvalue of  $u_1$ . Let  $\alpha \in \mathbb{F} \setminus \{0\}$ . Then,  $\text{Ker}(u_1 - \alpha \text{id}_{V_1}) \subset \text{Im } u_1$ , and hence the codimension of  $\text{Ker}(u_1 - \alpha \text{id}_{V_1})$  in  $V_1$  is greater than or equal to the dimension of  $W'$ , which proves that  $\alpha$  is not a dominant eigenvalue of  $u_1$ .

Therefore,  $u_1$  has no dominant eigenvalue.  $\square$

From there, we can derive Theorem 4 from Theorem 2. Assume that Theorem 2 is valid. Let  $p_1, p_2, p_3$  be split polynomials with degree 2 over  $\mathbb{F}$ , and let  $u$  be an endomorphism of an infinite-dimensional vector space  $V$ . Assume that  $u$  has a dominant eigenvalue  $\lambda$ , that  $u - \lambda \text{id}_V$  has infinite rank and that either  $\lambda$  is a  $(p_1, p_2, p_3)$ -sum or  $2\lambda = \text{tr } p_1 + \text{tr } p_2 + \text{tr } p_3$ .

We can find a decomposition  $V = V_1 \oplus V_2$  given by Lemma 15. Then, the endomorphism of  $V_2$  induced by  $u$  is a  $(p_1, p_2, p_3)$ -sum, owing to Proposition 12. On the other hand, the endomorphism  $u_1$  of  $V_1$  induced by  $u$  has no dominant eigenvalue and  $V_1$  is infinite-dimensional, whence by Theorem 2 the endomorphism  $u_1$  is a  $(p_1, p_2, p_3)$ -sum. By Remark 1, we conclude that  $u$  is a  $(p_1, p_2, p_3)$ -sum.

## 6 Stratifications

Throughout this section, we shall need the following notation on well-ordered sets:

**Definition 4.** Let  $D$  be a well-ordered set and  $\alpha$  be an element of  $D$ , but not the greatest one. Then, we denote by  $\alpha + 1$  the **successor** of  $\alpha$  (that is, the least element of  $\{\beta \in D : \alpha < \beta\}$ ). We say that  $\alpha$  has a predecessor whenever  $\alpha$  is the successor of some element of  $D$ .

### 6.1 A review of known results

**Definition 5.** Let  $V$  be a non-zero  $\mathbb{F}[t]$ -module. A **stratification** of  $V$  is an increasing sequence  $(V_\alpha)_{\alpha \in D}$ , indexed over a well-ordered set  $D$ , of submodules of  $V$  in which:

- For all  $\alpha \in D$ , the quotient module  $V_\alpha / \left( \sum_{\beta < \alpha} V_\beta \right)$  is non-zero and monogenous;
- $V = \sum_{\alpha \in D} V_\alpha$ .

To any such stratification, we assign the **dimension sequence**  $(n_\alpha)_{\alpha \in D}$  defined by

$$n_\alpha := \dim_{\mathbb{F}} \left( V_\alpha / \sum_{\beta < \alpha} V_\beta \right)$$

(if this dimension is not finite, we shall denote it by  $+\infty$  rather than by  $\aleph_0$ ).

Let  $(V_\alpha)_{\alpha \in D}$  be a stratification of  $V$ . For every  $\alpha \in D$ , we can choose a vector  $x_\alpha \in V_\alpha$  such that  $V_\alpha = \mathbb{F}[t]x_\alpha + \sum_{\beta < \alpha} V_\beta$ , and we note that if  $n_\alpha$  is finite then  $V_\alpha = \mathbb{F}_{n_\alpha-1}[t]x_\alpha \oplus \sum_{\beta < \alpha} V_\beta$ , otherwise  $V_\alpha = \mathbb{F}[t]x_\alpha \oplus \sum_{\beta < \alpha} V_\beta$ , and in any case  $(t^k x_\alpha)_{0 \leq k < n_\alpha}$  is linearly independent over  $\mathbb{F}$ . We shall say that the **vector sequence**  $(x_\alpha)_{\alpha \in D}$  is attached to  $(V_\alpha)_{\alpha \in D}$ . In that case, an obvious transfinite induction shows that, for all  $\alpha$  and  $\beta$  in  $D$  with  $\beta < \alpha$ , the family  $(t^k x_\delta)_{\beta \leq \delta \leq \alpha, 0 \leq k < n_\delta}$  is linearly independent over  $\mathbb{F}$  and

$$\left[ \sum_{\gamma < \beta} V_\gamma \right] \oplus \text{span}_{\mathbb{F}} \left( (t^k x_\delta)_{\beta \leq \delta \leq \alpha, 0 \leq k < n_\delta} \right) = V_\alpha.$$

Moreover the family  $(t^k x_\delta)_{\beta \leq \delta < \alpha, 0 \leq k < n_\delta}$  is linearly independent over  $\mathbb{F}$  and

$$\left[ \sum_{\gamma < \beta} V_\gamma \right] \oplus \text{span}_{\mathbb{F}} \left( (t^k x_\delta)_{\beta \leq \delta < \alpha, 0 \leq k < n_\delta} \right) = \sum_{\gamma < \alpha} V_\gamma.$$

In particular,  $(t^k x_\alpha)_{\alpha \in D, 0 \leq k < n_\alpha}$  is a basis of the vector space  $V$ . As a special case, we get the obvious consequence:

**Lemma 16.** *Let  $V$  be an  $\mathbb{F}[t]$ -module with a stratification  $(V_\alpha)_{\alpha \in D}$  whose corresponding dimension sequence is denoted by  $(n_\alpha)_{\alpha \in D}$ . Assume that  $n_\alpha = +\infty$  for all  $\alpha \in D$ . Then,  $V$  is free.*

Conversely, consider a sequence  $(x_\alpha)_{\alpha \in D}$ , indexed over a well-ordered set  $D$ , of vectors of  $V$  such that  $x_\alpha \notin \sum_{\beta < \alpha} \mathbb{F}[t]x_\beta$  for all  $\alpha \in D$ , and  $V = \sum_{\alpha \in D} \mathbb{F}[t]x_\alpha$ . Then, one sees that  $\left( \sum_{\beta \leq \alpha} \mathbb{F}[t]x_\beta \right)_{\alpha \in D}$  is a stratification of  $V$  with corresponding vector sequence  $(x_\alpha)_{\alpha \in D}$ .

Let us now recall the definition of a connector.

**Definition 6.** Let  $u$  be an endomorphism of a vector space  $V$ . Let  $(V_\alpha)_{\alpha \in D}$  be a stratification of  $V^u$ , with attached dimension sequence  $(n_\alpha)_{\alpha \in D}$  and an associated vector sequence  $(x_\alpha)_{\alpha \in D}$ .

An endomorphism  $v$  of  $V$  is called a **connector** for  $u$  with respect to the vector sequence  $(x_\alpha)_{\alpha \in D}$  whenever it acts as follows on the basis  $(t^k x_\alpha)_{\alpha \in D, 0 \leq k < n_\alpha}$ : for all  $\alpha \in D$  such that  $n_\alpha < +\infty$  and  $\alpha$  is not the greatest element of  $D$ , we have  $v(t^{n_\alpha-1} x_\alpha) = x_{\alpha+1}$  modulo  $V_\alpha$ , and all the other vectors are mapped to 0.

Here is the basic result that demonstrates the interest of connectors:

**Proposition 17** (Proposition 8 of [5]). *Let  $u$  be an endomorphism of a vector space  $V$ . Let  $(V_\alpha)_{\alpha \in D}$  be a stratification of  $V^u$ , with attached dimension sequence  $(n_\alpha)_{\alpha \in D}$  and an associated vector sequence  $(x_\alpha)_{\alpha \in D}$ .*

*Assume that if  $D$  has a maximum  $M$  then  $n_M = +\infty$ . Then, for any connector  $v$  for  $u$  with respect to  $(x_\alpha)_{\alpha \in D}$ , the endomorphism  $u + v$  is elementary.*

## 6.2 Good stratifications

In light of Proposition 17, our wish is to create a connector that is annihilated by  $t^2 - at$  for a given scalar  $a$ . This is possible if we consider special cases of stratifications:

**Definition 7.** Let  $(V_\alpha)_{\alpha \in D}$  be a stratification of  $V$ , with corresponding dimension sequence  $(n_\alpha)_{\alpha \in D}$ . We define three potential properties of that stratification:

- (A) One has  $n_\alpha \geq 2$  whenever  $\alpha \in D$  has a predecessor.
- (A<sup>+</sup>) One has  $n_\alpha \geq 2$  whenever  $\alpha \in D$  has a predecessor or  $\alpha$  is the minimum of  $D$ .
- (M) There is no maximum in  $D$ .

A stratification is called **good** whenever it satisfies both properties (A<sup>+</sup>) and (M).

The following basic result motivates that we focus on good stratifications:

**Proposition 18.** *Let  $V$  be a vector space and  $u$  be an endomorphism of  $V$ . Let  $a \in \mathbb{F}$ . Let  $(V_\alpha)_{\alpha \in D}$  be a stratification of  $V^u$  that satisfies properties (A) and (M). Then,  $u$  is  $a$ -elementarily decomposable.*

*Proof.* Let  $(x_\alpha)_{\alpha \in D}$  be a vector sequence attached to  $(V_\alpha)_{\alpha \in D}$ , and denote by  $(n_\alpha)_{\alpha \in D}$  the associated dimension sequence. We define  $v \in \text{End}(V)$  on the basis  $(u^k(x_\alpha))_{\alpha \in D, 0 \leq k < n_\alpha}$  as follows: For all  $\alpha \in D$  such that  $n_\alpha < +\infty$ , we put

$$v(u^{n_\alpha-1}(x_\alpha)) := au^{n_\alpha-1}(x_\alpha) - x_{\alpha+1},$$

(note that this makes sense because, by property (M), the element  $\alpha$  must have a successor) and all the other basis vectors are mapped to 0. Then,  $-v$  is a connector for  $u$  with respect to the sequence  $(x_\alpha)_{\alpha \in D}$ , and hence  $u - v$  is elementary. On the other hand, we check that  $v^2 = av$ : Given  $\alpha \in D$  such that  $n_\alpha < +\infty$ , we see that  $v(x_{\alpha+1}) = 0$  because of property (A), and it follows that  $v^2$  and  $av$  agree on  $u^{n_\alpha-1}(x_\alpha)$ ; on the other hand both  $v^2$  and  $av$  vanish at all the other basis vectors, and hence  $v^2 = av$ . This completes the proof.  $\square$

**Corollary 19.** *Let  $V$  be a vector space and  $u$  be an endomorphism of  $V$  with a good stratification. Then, for all  $a \in \mathbb{F}$ , the endomorphism  $u$  is  $a$ -elementarily decomposable.*



### 6.3 A technical lemma on special stratifications

The following technical result will be used in remote parts of the article.

**Lemma 20.** *Let  $V$  be a non-zero  $\mathbb{F}[t]$ -module. Assume that there is a non-zero submodule  $W$  of  $V$  such that  $V/W$  has a good stratification and  $W$  has a stratification that satisfies  $(A^+)$ . Then,  $V$  has a good stratification.*

*Proof.* We take a stratification  $(W_k)_{k \in D}$  of  $W$  that satisfies  $(A^+)$  and a good stratification  $(V_l)_{l \in D'}$  of  $V/W$ , in which  $D$  and  $D'$  are ordinals. We denote by  $(n_k)_{k \in D}$  and  $(m_l)_{l \in D'}$  the associated dimension sequences. We equip

$$L := (\{0\} \times D) \cup (\{1\} \times D')$$

with the lexicographic ordering, which makes it a well-ordered set. For  $k \in D$ , we set  $E_{0,k} := W_k$  and, for  $l \in D'$ , we define  $E_{1,l}$  as the inverse image of  $V_l$  under the canonical projection from  $V$  to  $V/W$ . In particular,  $E_{0,k} \subset W \subsetneq E_{1,l}$  for all  $k \in D$  and  $l \in D'$ , and it is easily checked that  $(E_a)_{a \in L}$  is an increasing sequence of submodules of  $V$ . Moreover, for all  $k \in D$ , we see that

$\sum_{a \in L, a < (0,k)} E_a = \sum_{k' \in D, k' < k} W_{k'}$  and hence the  $\mathbb{F}[t]$ -module  $E_{0,k} / \left( \sum_{a \in L, a < (0,k)} E_a \right)$  is monogenous and has dimension  $n_k$  as an  $\mathbb{F}$ -vector space. Given  $l \in D'$ , since  $(E_a)_{a \in L}$  is increasing we see that  $\sum_{a \in L, a < (1,l)} E_a = W + \sum_{l' \in D', l' < l} E_{0,l'}$  which includes

$W$ , and hence

$$E_{1,l} / \left( \sum_{a \in L, a < (1,l)} E_a \right) = E_{1,l} / \left( W + \sum_{l' \in D', l' < l} E_{0,l'} \right) \simeq V_l / \left( \sum_{l' \in D', l' < l} V_{l'} \right).$$

Therefore, the  $\mathbb{F}[t]$ -module  $E_{1,l} / \left( \sum_{a \in L, a < (1,l)} E_a \right)$  is monogenous and has dimension  $m_l$  as an  $\mathbb{F}$ -vector space.

Hence,  $(E_a)_{a \in L}$  is a stratification of  $V$ . It remains to check that it is a good one. If  $L$  had a maximum, then this maximum would read  $(1, M)$  and  $M$  would be the maximum of  $D'$ , which is impossible because  $(V_l)_{l \in D'}$  has property (M).

Finally, let  $a \in L$  have a predecessor in  $L$  or be the minimum of  $L$ . If  $a = (0, k)$  for some  $k \in D$ , then  $k$  has a predecessor in  $D$  or is the minimum of  $D$ , and we deduce that  $n_k \geq 2$ . If  $a = (1, l)$  for some  $l \in D'$  then we obtain likewise that  $m_l \geq 2$ . We deduce that  $(E_a)_{a \in L}$  has property  $(A^+)$ .  $\square$

## 7 Endomorphisms with no dominant eigenvalue: The uncountable-dimensional case

Here, we consider the case of a vector space with uncountable dimension. In order to prove Theorem 11 in that restricted setting, we know from Corollary 19 that it suffices to prove the following result:

**Proposition 21.** *Let  $V$  be a vector space with uncountable dimension, and  $u$  be an endomorphism of  $V$  with no dominant eigenvalue. Then,  $V^u$  has a good stratification.*

*Proof.* Denote by  $\kappa$  the dimension of  $V$ : it is a cardinal. Since  $\kappa$  is uncountable, the set  $L$  consisting of the limit ordinals in  $\kappa$  has cardinality  $\kappa$ , whence we can choose a basis  $(e_\alpha)_{\alpha \in L}$  of the  $\mathbb{F}$ -vector space  $V$ .

The construction is now done by transfinite induction. Let  $\alpha \in \kappa$ , and assume that we have constructed partial sequences  $(E_\beta)_{\beta < \alpha}$ ,  $(x_\beta)_{\beta < \alpha}$  and  $(n_\beta)_{\beta < \alpha}$  of, respectively, linear subspaces, nonzero vectors, and elements of  $\mathbb{N}^* \cup \{+\infty\}$ , such that:

- (i) For all  $\beta < \alpha$ ,  $E_\beta$  is a linear subspace of  $V$  and  $(u^k(x_\beta))_{0 \leq k < n_\beta}$  is a basis of it;
- (ii) The vector spaces  $E_\beta$ , for  $\beta < \alpha$ , are linearly disjoint;
- (iii) For all  $\beta < \alpha$ , if  $n_\beta < +\infty$  then  $u^{n_\beta}(x_\beta) \in \bigoplus_{\gamma \leq \beta} E_\gamma$ ;
- (iv) For every  $\beta < \alpha$ , if  $\beta \notin L$  then  $n_\beta \geq 2$ , otherwise  $e_\beta \in \sum_{\gamma \leq \beta} E_\gamma$ .

Set  $W := \bigoplus_{\beta < \alpha} E_\beta$ . By properties (i) and (iii), the linear subspace  $W$  is stable under  $u$ .

We claim that the endomorphism  $\bar{u}$  induced by  $u$  on  $V/W$  is not a scalar multiple of identity. If it were, there would be a scalar  $\lambda$  such that  $\text{Im}(u - \lambda \text{id}_V) \subset W$ . However, since  $\kappa$  is a cardinal,  $\{\beta \in \kappa : \beta < \alpha\}$  has its cardinality less than  $\kappa$ , and since each  $E_\beta$  has its dimension countable or finite, this yields  $\dim W < \kappa$ . Hence,  $\lambda$  would be a dominant eigenvalue of  $u$ , in contradiction with our assumptions. By the classical characterization of the scalar multiples of the identity among the endomorphisms, there exists a vector  $y \in V$  such that  $(y, u(y))$  is linearly independent modulo  $W$ .

Now, we put  $x_\alpha := e_\alpha$  if  $\alpha \in L$  and  $e_\alpha \notin W$ , otherwise  $x_\alpha := y$ . In any case, we take  $n_\alpha$  as the supremum of the set of all  $k \in \mathbb{N}$  for which  $(u^i(x_\alpha))_{0 \leq i < k}$  is linearly independent modulo  $W$ , and

$$E_\alpha := \text{span}(u^i(x_\alpha))_{0 \leq i < n_\alpha}.$$

By the very definition of  $x_\alpha$ , we have  $n_\alpha \geq 2$  if  $\alpha \notin L$ , and  $x_\alpha \in \sum_{\beta \leq \alpha} E_\beta$  otherwise.

It is then easily checked that the spaces  $E_\beta$ , for  $\beta \leq \alpha$ , are linearly disjoint and that if  $n_\alpha < +\infty$  then  $u^{n_\alpha}(x_\alpha) \in \bigoplus_{\beta \leq \alpha} E_\beta$ .

The inductive step is now achieved. By property (iv) above, the subspace  $\sum_{\beta \in \kappa} E_\beta$  contains all the basis vectors  $e_\alpha$  with  $\alpha \in L$ , and hence  $V = \sum_{\beta \in \kappa} E_\beta$ . For  $\alpha \in \kappa$ , set  $V_\alpha := \bigoplus_{\beta \leq \alpha} E_\beta$ . Then, one sees from properties (i) to (iv) that  $(V_\alpha)_{\alpha \in \kappa}$  is a good stratification of  $V^u$ .  $\square$

To further illustrate the specificity of the uncountable-dimensional case, we give an example when  $V^u$  has no good stratification whereas  $u$  has no dominant eigenvalue.

*Example 4.* Consider the  $\mathbb{F}[t]$ -module  $V := \mathbb{F}[t] \times (\mathbb{F}[t]/(t))$ , and consider the vectors  $e := (1, 0)$  and  $f := (0, 1)$  in  $V$ , so that  $V = \mathbb{F}[t]e \oplus \mathbb{F}[t]f$ . Assume that  $V$  has a good stratification  $(V_\alpha)_{\alpha \in D}$ , and denote by  $m$  the least element of  $D$ . Let  $x$  be a generator of  $V_m$ . Since  $\dim V_m \geq 2$ , we have  $x \notin \mathbb{F}[t]f$ . Hence,  $x = p(t)e + \lambda f$  for some  $p(t) \in \mathbb{F}[t] \setminus \{0\}$  and some  $\lambda \in \mathbb{F}$ . The degree  $d$  of  $p(t)$  is non-negative, and it is easily checked that  $V_m = \mathbb{F}[t]x$  does not contain  $f$ . Hence,  $m$  has a successor in  $D$ , which we denote by  $m+1$ . Moreover, it is obvious that the respective classes of  $f, e, te, \dots, t^{d-1}e$  generate the vector space  $V/V_m$ , and hence  $V/V_m$  is a non-zero  $\mathbb{F}$ -vector space with finite dimension. Hence,  $D$  must be finite, and  $(V_\alpha)_{\alpha \in D}$  fails to be a good stratification since it does not satisfy condition (M).

With a similar strategy, one can prove that  $\mathbb{F}[t] \times (\mathbb{F}[t]/(t))^2$  has no stratification that satisfies both conditions (A) and (M).

## 8 Endomorphisms with no dominant eigenvalue: The countable-dimensional case

In this section, we shall complete the proof of Theorem 11 by tackling the special case of vector spaces with countable dimension. Here, the situation is far

more complicated than the one of the preceding section because, for an endomorphism  $u$  with no dominant eigenvalue, the module  $V^u$  might not have a good stratification (see Example 4).

We shall start by considering the case when  $V^u$  is a torsion  $\mathbb{F}[t]$ -module, and we will show in this situation (and still assuming that  $u$  has no dominant eigenvalue) that it must have a good stratification (Section 8.1). In Section 8.2, we will complete the proof by tackling the case when  $V^u$  is not a torsion  $\mathbb{F}[t]$ -module, with the help of some results from the torsion case.

### 8.1 The case of a torsion $\mathbb{F}[t]$ -module

Our aim is to prove the following result:

**Proposition 22.** *Let  $u$  be an endomorphism of a vector space  $V$  with countable dimension. Assume that  $u$  has no dominant eigenvalue and that  $V^u$  is a torsion  $\mathbb{F}[t]$ -module. Then,  $V^u$  has a good stratification.*

Combining this result with Corollary 19 obviously yields Theorem 11 in the special case when  $V^u$  has countable dimension over  $\mathbb{F}$  and is a torsion module.

For its proof, we need two lemmas.

**Lemma 23.** *Let  $u$  be an endomorphism of a vector space  $V$  with countable dimension. Assume that  $u$  has no dominant eigenvalue and that  $(u - \lambda \text{id}_V)^2 = 0$  for some  $\lambda \in \mathbb{F}$ . Then,  $V^u$  has a good stratification.*

*Proof.* Since  $(u - \lambda \text{id}_V)^2 = 0$ , there are families  $(e_i)_{i \in I}$  and  $(f_j)_{j \in J}$  of non-zero vectors such that

$$V = \bigoplus_{i \in I} \text{span}(e_i) \oplus \bigoplus_{j \in J} \text{span}(f_j, u(f_j)),$$

with  $e_i \in \text{Ker}(u - \lambda \text{id}_V)$  for all  $i \in I$ , and  $f_j \notin \text{Ker}(u - \lambda \text{id}_V)$  for all  $j \in J$ .

Since  $\lambda$  is not a dominant eigenvalue of  $u$ , the set  $J$  is infinite, and hence countable. We choose a subset  $A$  of  $\mathbb{N} \setminus \{0\}$  with the same cardinality as  $I$ , and we put  $I' := A \times \{0\}$  and  $J' := \mathbb{N}^2 \setminus I'$ , so that  $J'$  is equipotent to  $J$ . Hence, without loss of generality we can assume that  $(I, J)$  is a partition of  $\mathbb{N}^2$  and  $I \subset (\mathbb{N} \setminus \{0\}) \times \{0\}$ . Then, for  $(k, l) \in \mathbb{N}^2$ , we put  $z_{k,l} := e_{k,l}$  if  $(k, l) \in I$ , and  $z_{k,l} := f_{k,l}$  otherwise. The set  $\mathbb{N}^2$  is well-ordered by the lexicographic ordering. Then, one checks that the vector sequence  $(z_{k,l})_{(k,l) \in \mathbb{N}^2}$  defines a good stratification of  $V^u$ .  $\square$

**Lemma 24.** *Let  $u$  be an endomorphism of a vector space  $V$  with infinite dimension. Assume that  $u$  has no dominant eigenvalue, that there is no scalar  $\lambda$  such that  $(u - \lambda \text{id}_V)^2 = 0$ , and that  $V^u$  is a torsion  $\mathbb{F}[t]$ -module. Let  $x_0$  be a vector of  $V$ . Then, there are submodules  $V_0$  and  $V_1$  of  $V^u$  such that:*

- (i)  $V_0 \subset V_1$ ;
- (ii)  $V_1$  contains  $x_0$ ;
- (iii)  $\dim_{\mathbb{F}} V_0 > 1$  and  $\dim_{\mathbb{F}}(V_1/V_0) > 1$ ;
- (iv) Each module  $V_0$  and  $V_1/V_0$  is monogenous.

*Proof.* We distinguish between several cases.

**Case 1.**  $x_0$  is not an eigenvector of  $u$ .

Then, we set  $V_0 := \mathbb{F}[t]x_0$ , which is finite-dimensional as a vector space. Note that the endomorphism of  $V/V_0$  induced by  $u$  has no dominant eigenvalue, whence some non-zero vector  $z \in V/V_0$  is not an eigenvector of it. Denote by  $V_1$  the inverse image of  $\mathbb{F}[t]z$  under the canonical projection  $V \rightarrow V/V_0$ . Then,  $V_1/V_0 = \mathbb{F}[t]z$  and  $V_1/V_0$  has finite dimension greater than 1 as a vector space over  $\mathbb{F}$ .

**Case 2.**  $x_0$  is an eigenvector of  $u$ .

Without loss of generality, we can assume that  $u(x_0) = 0$ .

**Case 2.1.** The endomorphism  $u$  is not locally nilpotent, i.e. we do not have  $\forall x \in V, \exists n \in \mathbb{N} : u^n(x) = 0$ .

Since  $V^u$  is a torsion module, there is a non-zero vector  $y$  together with a monic irreducible polynomial  $p(t) \neq t$  such that  $p(u)[y] = 0$ . Then,  $t$  and  $p(t)$  are coprime, whence  $V_0 := \mathbb{F}[t](x_0 + y)$  contains  $x_0$  and  $y$ , and in particular  $\dim_{\mathbb{F}} V_0 \geq 2$ . Then, we find a submodule  $V_1 \supset V_0$  as in Case 1.

**Case 2.2.** The endomorphism  $u$  is locally nilpotent.

Our assumptions tell us that  $u^2 \neq 0$ . This yields a vector  $y \in V$  such that  $u^2(y) \neq 0$  and  $u^3(y) = 0$ . Set  $F := \text{span}(y, u(y), u^2(y), x_0)$ .

**Case 2.2.1.** One has  $x_0 \in \text{span}(y, u(y), u^2(y))$ .

Then, we set  $V_0 := \mathbb{F}[t]y = F$ , and we construct  $V_1$  as in Case 1.

**Case 2.2.2.** One has  $x_0 \notin \text{span}(y, u(y), u^2(y))$ .

Then, we set  $V_0 := \text{span}(u(y) + x_0, u^2(y))$  and  $V_1 := F$ . Note that  $V_0$  is the  $\mathbb{F}[t]$ -submodule generated by  $u(y) + x_0$ , and that  $(\overline{x_0}, \overline{y})$  is a basis of the quotient space  $V_1/V_0$ . Noting that  $u(y) = -x_0$  modulo  $V_0$ , we see that the  $\mathbb{F}[t]$ -module  $V_1/V_0$  is generated by  $\overline{y}$ . Hence,  $V_0$  and  $V_1$  have the expected properties.  $\square$

Now, we can prove Proposition 22.

*Proof of Proposition 22.* By a *reductio ad absurdum*, we assume that  $V^u$  has no good stratification. We choose a basis  $(e_n)_{n \in \mathbb{N}}$  of the  $\mathbb{F}$ -vector space  $V$ .

Then, by induction, we shall construct a good stratification of  $V^u$ . Set  $V_{-1} := \{0\}$ . Let  $n \in \mathbb{N}$ , and assume that we have constructed a partial increasing sequence  $(V_k)_{0 \leq k \leq 2n-1}$  of submodules of  $V^u$  such that  $V_k/V_{k-1}$  is finite-dimensional over  $\mathbb{F}$  with dimension greater than 1 for all  $k \in \llbracket 0, n-1 \rrbracket$ , and  $V_{2k+1}$  contains  $e_k$  for all  $k \in \llbracket 0, n-1 \rrbracket$ .

Then, we consider the quotient vector space  $V/V_{2n-1}$  and the induced endomorphism  $\bar{u}$  of it. Since  $V_{2n-1}$  is finite-dimensional as a vector space and  $u$  has no dominant eigenvalue,  $\bar{u}$  has no dominant eigenvalue either. Moreover,  $V/V_{2n-1}$  is a torsion  $\mathbb{F}[t]$ -module.

Assume first that there is a scalar  $\lambda$  such that  $(\bar{u} - \lambda \text{id})^2 = 0$ . Then, by Lemma 23, there is a good stratification of  $V/V_{2n-1}$ . However,  $(V_k)_{0 \leq k \leq 2n-1}$  is obviously a stratification of  $V_{2n-1}$  with property  $(A^+)$ . Hence, by Lemma 20, there is a good stratification of  $V$ , contradicting our assumptions.

Hence, there is no scalar  $\lambda$  such that  $(\bar{u} - \lambda \text{id})^2 = 0$ . By Lemma 24, there are submodules  $W_0 \subset W_1$  of  $V/V_{2n-1}$  such that  $W_1$  contains the class of  $e_n$  modulo  $V_{2n-1}$ , and both modules  $W_0$  and  $W_1/W_0$  are monogenous and have their dimension over  $\mathbb{F}$  finite and greater than 1. Then, we define  $V_{2n}$  and  $V_{2n+1}$  as the respective inverse images of  $W_0$  and  $W_1$  under the canonical projection of  $V$  onto  $V/V_{2n-1}$ . Then,  $V_{2n+1}/V_{2n}$  and  $V_{2n}/V_{2n-1}$  are isomorphic to, respectively,  $W_1/W_0$  and  $W_0$ , and hence both are monogenous and have their dimension over  $\mathbb{F}$  finite and greater than 1. Finally,  $V_{2n+1}$  contains  $e_n$ .

Hence, by induction we have constructed an increasing sequence  $(V_n)_{n \in \mathbb{N}}$  of submodules of  $V$  such that each quotient module  $V_n/V_{n-1}$  is monogenous and has its dimension over  $\mathbb{F}$  finite and greater than 1, and  $V_{2n+1}$  contains  $e_n$  for all  $n \in \mathbb{N}$ . The latter property yields  $\sum_{n \in \mathbb{N}} V_n = V$ , and we deduce that  $(V_n)_{n \in \mathbb{N}}$  is a good stratification of  $V^u$ . This completes the proof.  $\square$

We finish this section with a basic result on the case of a dominant eigenvalue, to be used in the non-torsion case.

**Lemma 25.** *Let  $u$  be an endomorphism of a vector space  $V$ , and assume that there is a scalar  $\lambda$  such that  $u - \lambda \text{id}_V$  has finite rank. Then,  $V^u$  splits into  $V^u = W \oplus H$  in which:*

- $W$  is a finite direct sum of monogenous submodules with dimension over  $\mathbb{F}$  finite and greater than 1;
- There is a scalar  $\mu$  such that  $\forall x \in H, u(x) = \mu x$ .

*Proof.* If  $V$  is finite-dimensional, the result is an obvious consequence of the classification of finitely generated torsion  $\mathbb{F}[t]$ -modules. In the rest of the proof, we assume that  $V$  is infinite-dimensional. Set  $w := u - \lambda \text{id}_V$ .

Let us choose a finite-dimensional linear subspace  $W'$  of  $V$  such that  $\text{Im } w \subset W'$  and  $W' + \text{Ker } w = V$ . Let us choose a complementary subspace  $H'$  of  $W'$  in  $V$  such that  $H' \subset \text{Ker } w$ . Note that  $\forall x \in H', u(x) = \lambda x$ . Since  $W'$  includes  $\text{Im } w$  it is a submodule of  $V^u$ . By the classification of finitely generated  $\mathbb{F}[t]$ -modules, there is a scalar  $\mu$  together with a splitting

$$W' = E \oplus \bigoplus_{i=1}^n W_i$$

in which  $\forall x \in E, u(x) = \mu x$  and each submodule  $W_i$  is monogenous with (finite) dimension greater than 1.

Then, there are two cases to consider.

- If  $\mu = \lambda$  then we take  $H := E \oplus H'$  and  $W := \bigoplus_{i=1}^n W_i$ .
- Assume that  $\mu \neq \lambda$ . Then, we choose a basis  $(e_1, \dots, e_m)$  of the  $\mathbb{F}$ -vector space  $E$  and then a linearly independent  $m$ -tuple  $(f_1, \dots, f_m)$  of vectors of  $H'$ , and we re-split  $H' = \text{span}(f_1, \dots, f_m) \oplus H$  for some linear subspace  $H$ . Then, we set  $W := \bigoplus_{i=1}^n W_i \oplus \bigoplus_{i=1}^m \text{span}(e_i, f_i)$  and we note that  $\text{span}(e_i, f_i) = \mathbb{F}[t](e_i + f_i)$  is monogenous with dimension 2 for all  $i \in \llbracket 1, m \rrbracket$ .

□

## 8.2 The case of non-torsion $\mathbb{F}[t]$ -modules

**Definition 8.** Let  $V$  be an  $\mathbb{F}[t]$ -module and  $F$  be a free submodule of  $V$ . We say that  $F$  is **quasi-maximal** if  $V/F$  is a torsion module.

Equivalently,  $F$  is quasi-maximal if and only if there is no non-zero free submodule  $F'$  of  $V$  such that  $F \cap F' = \{0\}$ . Beware that a quasi-maximal free

submodule may not be maximal among the free submodules: for example, in the  $\mathbb{F}[t]$ -module  $\mathbb{F}[t]$ , the free submodule  $t\mathbb{F}[t]$  is quasi-maximal but it is not a maximal free submodule.

To construct a quasi-maximal free submodule of  $V$ , it suffices to take a maximal  $\mathbb{F}[t]$ -independent subset  $A$  of  $V$  (which exists thanks to Zorn's lemma) and to consider the free module  $\sum_{x \in A} \mathbb{F}[t]x$ .

In order to prove Theorem 11 in the remaining case when  $V^u$  is not a torsion module and the dimension of  $V$  is countable, the main step consists in the following decomposition of a non-torsion  $\mathbb{F}[t]$ -module devoid of a good stratification.

**Lemma 26.** *Let  $V$  be an  $\mathbb{F}[t]$ -module. Assume that  $V$  is not a torsion module, that  $V$  has countable dimension as a vector space over  $\mathbb{F}$ , and that  $V$  has no good stratification. Then, there exist submodules  $F \subset W$  of  $V$  together with a scalar  $\lambda$  such that:*

- (a)  $F$  is a non-zero free submodule of  $V$ ;
- (b)  $W/F$  has finite dimension over  $\mathbb{F}$  and, if nonzero, has a stratification that satisfies condition  $(A^+)$ ;
- (c) There is a splitting  $V = W \oplus H$  such that  $\forall x \in H, tx = \lambda x$ .

*Proof.* We start by choosing a quasi-maximal free submodule  $F'$  of  $V$ . Note that  $F' \neq \{0\}$  since  $V$  is not a torsion module. The quotient module  $V/F'$  is a torsion module whose dimension is at most countable. We can choose a basis  $(e_i)_{i \in I}$  of the free module  $F'$  indexed by a subset  $I$  of  $\mathbb{N}$ . Then, by setting  $V_i := \sum_{j \in I, j \leq i} \mathbb{F}[t]e_j$ , we see that  $(V_i)_{i \in I}$  is a good stratification of  $F'$ .

Hence, if  $V/F'$  had a good stratification, Lemma 20 would yield that  $V$  has a good stratification, which has been ruled out. If  $F' = V$  then  $V$  has a good stratification.

Hence,  $V/F'$  is nonzero and it has no good stratification. It follows from Proposition 22 that  $V/F'$  is finite-dimensional as a vector space over  $\mathbb{F}$  or the endomorphism  $x \mapsto tx$  of  $V/F'$  has a dominant eigenvalue. In any case, Lemma 25 yields a scalar  $\lambda$  and a module splitting

$$V/F' = K \oplus G$$

in which:



- Each vector of  $K$  is annihilated by  $t - \lambda$ ;
- The module  $G$  splits into a finite direct sum of monogenous submodules, each of which with dimension over  $\mathbb{F}$  finite and greater than 1.

In particular,  $G$  has a finite stratification that satisfies condition  $(A^+)$ .

Since  $p(t) \mapsto p(t + \lambda)$  is an automorphism of the algebra  $\mathbb{F}[t]$ , we lose no generality in assuming that  $\lambda = 0$ , and we shall assume that this condition holds throughout the remainder of the proof.

Next, we define  $V_1$  as the inverse image of  $K$  under the canonical projection of  $V$  onto  $V/F'$ . It follows that  $V/V_1$  is isomorphic to  $G$ , and hence it has a stratification that satisfies  $(A^+)$ .

Since  $F'$  is a free  $\mathbb{F}[t]$ -module, we can choose an  $\mathbb{F}$ -linear subspace  $F'_0$  of it such that  $F' = \bigoplus_{n \in \mathbb{N}} t^n F'_0$  and  $x \in F'_0 \mapsto t^n x$  is injective for all  $n \in \mathbb{N}$ .

Next, we consider the inverse image  $L$  of  $F'_0$  under  $x \in V_1 \mapsto tx$ . We have  $L + F' = V_1$ : indeed, given  $x \in V_1$ , we have  $tx \in F'$  and hence  $tx = x_0 + tx_1$  for some  $x_0 \in F'_0$  and some  $x_1 \in F'$ , whence  $x - x_1 \in L$ .

It follows that we can find a linear subspace  $H' \subset L$  such that

$$V_1 = F' \oplus H',$$

leading to  $\forall x \in H', tx \in F'_0$ . Next, we split  $H'$  as follows: we consider the linear mapping  $h : x \in H' \mapsto tx \in F'_0$ , we denote by  $H$  its kernel, and we consider a complementary subspace  $F_1$  of  $H$  in  $H'$ . It follows that  $\forall x \in H, tx = 0$ , whereas  $x \mapsto tx$  maps  $F_1$  bijectively onto a linear subspace  $F'_1$  of  $F'_0$ . Finally, we consider a complementary subspace  $F_0$  of  $F'_1$  in  $F'_0$ , and we set

$$F := \left( \bigoplus_{n \in \mathbb{N}} t^n F_0 \right) \oplus \left( \bigoplus_{n \in \mathbb{N}} t^n F_1 \right) = \mathbb{F}[t]F_0 \oplus \mathbb{F}[t]F_1 = \mathbb{F}[t]F_0 \oplus \mathbb{F}[t]F'_1 \oplus F_1 = F' \oplus F_1.$$

Then,  $F$  is a free submodule of  $V_1$  and

$$V_1 = F' \oplus H' = F' \oplus F_1 \oplus H = F \oplus H.$$

Moreover,  $F' \subset F \subset V_1$ .

Now, we choose a *linear subspace*  $G'$  of  $V$  which is mapped bijectively onto  $G$  under the canonical projection  $V \rightarrow V/F'$ . Then, since  $tx = 0$  for all  $x \in K$ , we know that  $tx \in F' + G' \subset F + G'$  for all  $x \in G'$ . Moreover, the definition of  $G'$  yields  $G' \cap V_1 = \{0\}$ , whence  $G' \cap F = \{0\}$ .

Hence,  $W := F \oplus G'$  is a submodule of  $V$  and  $W/F$  is isomorphic to  $G$ , which equals zero or has a stratification that satisfies condition  $(A^+)$ .

Since  $V/F' = K \oplus G$ , we have

$$V = V_1 \oplus G' = F \oplus H \oplus G' = W \oplus H,$$

which completes the proof.  $\square$

We conclude that, in order to establish Theorem 11 in the case of a countable-dimensional space and a non-torsion  $\mathbb{F}[t]$ -module, it only remains to prove the following result:

**Proposition 27.** *Let  $u$  be an endomorphism of a vector space  $V$  with countable dimension. Let  $\lambda \in \mathbb{F}$  and  $a \in \mathbb{F}$ . Assume that we have a splitting  $V^u = W \oplus H$  and a non-zero free submodule  $F$  of  $W$  such that:*

- (a) *The  $\mathbb{F}[t]$ -module  $W/F$  is finite-dimensional as an  $\mathbb{F}$ -vector space, and if nonzero it has a stratification that satisfies condition  $(A^+)$ .*
- (b)  *$\forall x \in H, u(x) = \lambda x$ .*

*Then,  $u$  is  $a$ -elementarily decomposable.*

To prove this result, the key is to consider the most simple situation, in which  $W = F$  and  $F$  is monogenous:

**Lemma 28** (Sewing lemma). *Let  $V$  be a vector space with countable dimension over  $\mathbb{F}$ , and  $u$  be an endomorphism of  $V$ . Assume that we have a module splitting  $V^u = V_1 \oplus V_2$  in which:*

- *$V_1$  is free, non-zero and monogenous;*
- *$u$  induces a scalar multiple of the identity on  $V_2$ .*

*Let  $x$  be a generator of  $V_1$  and  $a$  be a scalar. Then, there exists an endomorphism  $v$  of  $V$  such that  $v^2 = av$  and, for  $u' := u - v$ , one has  $V = \text{span}((u')^k(x))_{k \in \mathbb{N}}$ .*

In short, we have a very specific perturbation of  $u$  so as to turn  $V$  into the free monogenous  $\mathbb{F}[t]$ -module generated by  $x$ .

*Proof.* For  $n \in \mathbb{N}$ , set  $e_n := u^n(x)$ .

Assume first that  $V_2$  has countable dimension, and choose a basis  $(f_n)_{n \in \mathbb{N}}$  of  $V_2$ .

If  $a = 0$ , we define  $v$  on the basis  $(e_n) \amalg (f_n)$  as follows: for all  $n \in \mathbb{N}$ , we set

$$\begin{cases} v(e_{3n}) = e_{3n+1} - f_n \\ v(e_{3n+1}) = -e_{3n+1} + f_n \\ v(e_{3n+2}) = 0 \\ v(f_n) = -e_{3n+1} + f_n. \end{cases}$$

Otherwise, we define it on the same basis as follows: for all  $n \in \mathbb{N}$ , we set

$$\begin{cases} v(e_{3n}) = e_{3n+1} - f_n \\ v(e_{3n+1}) = 0 \\ v(e_{3n+2}) = 0 \\ v(f_n) = -ae_{3n+1} + af_n. \end{cases}$$

In any case, one easily checks that  $v^2 = av$ . Moreover, in any case, one also checks by induction on  $n$  that  $\text{span}((u-v)^k(x))_{0 \leq k \leq 4n} = \text{span}(e_0, e_1, \dots, e_{3n}, f_0, \dots, f_{n-1})$  for all  $n \in \mathbb{N}$ . This proves the claimed statement.

Assume finally that  $V_2$  has finite dimension  $p$ . If  $p = 0$ , we simply take  $v = 0$ . Now, assuming otherwise we choose a basis  $(f_0, \dots, f_{p-1})$  of  $V_2$ . Then, we slightly modify the above definition of  $v$ :

- If  $a = 0$ , we define, for all every non-negative integer  $n \leq p-1$ ,

$$\begin{cases} v(e_{3n}) = e_{3n+1} - f_n \\ v(e_{3n+1}) = -e_{3n+1} + f_n \\ v(e_{3n+2}) = 0 \\ v(f_n) = -e_{3n+1} + f_n \end{cases}$$

and for every integer  $k \geq 3p$  we set  $v(e_k) = 0$ ;

- If  $a \neq 0$ , we define, for all every non-negative integer  $n \leq p-1$ ,

$$\begin{cases} v(e_{3n}) = e_{3n+1} - f_n \\ v(e_{3n+1}) = 0 \\ v(e_{3n+2}) = 0 \\ v(f_n) = -ae_{3n+1} + af_n \end{cases}$$

and for every integer  $k \geq 3p$  we set  $v(e_k) = 0$ .

In any case, it is once more easy to check that  $v^2 = av$ . Moreover, one proves by finite induction that  $\text{span}((u-v)^k(x))_{0 \leq k \leq 4n} = \text{span}(e_0, e_1, \dots, e_{3n}, f_0, \dots, f_{n-1})$  for all  $n \in \llbracket 0, p \rrbracket$ , and then  $\text{span}((u-v)^k(x))_{0 \leq k \leq q} = \text{span}(e_0, e_1, \dots, e_{q-p}, f_0, \dots, f_{p-1})$  for all  $q \geq 4p$ . Again, the claimed statement is proved in that case.  $\square$

*Proof of Proposition 27.* We split  $F = \mathbb{F}[t]x \oplus F'$  for some non-zero vector  $x$  and some free submodule  $F'$ .

We assume first that  $F \subsetneq W$ . Let us consider a stratification  $(W_1, \dots, W_N)$  of  $W/F$  with property  $(A^+)$ , with associated dimension sequence  $(n_1, \dots, n_N)$  and an associated vector sequence  $(\overline{x_1}, \dots, \overline{x_N})$  in which  $x_i$  denotes a vector of  $W$  and  $\overline{x_i}$  denotes its class modulo  $F$ . We denote by  $M$  the dimension of the  $\mathbb{F}$ -vector space  $W/F$  and we set

$$\mathbf{B} := (x_1, \dots, u^{n_1-1}(x_1), x_2, \dots, u^{n_2-1}(x_2), \dots, u^{n_N-2}(x_N)) \quad \text{and} \quad G_1 := \text{span}(\mathbf{B}).$$

For all  $i \in \llbracket 1, N \rrbracket$ , there is a polynomial  $p_i(t) \in \mathbb{F}[t]$  such that  $u^{n_i}(x_i)$  equals  $p_i(t)x$  modulo  $F' + \text{span}(x_1, \dots, u^{n_1-1}(x_1), x_2, \dots, u^{n_2-1}(x_2), \dots, u^{n_i-1}(x_i))$ . We set

$$m := \max(0, \deg(p_1(t)), \dots, \deg(p_N(t))) \quad \text{and} \quad d := M + m.$$

Then, we consider the linear map

$$f : G_1 \oplus \mathbb{F}_{d-1}[t]x \oplus F' \rightarrow V$$

that sends  $u^{n_k-1}(x_k)$  to  $au^{n_k-1}(x_k) - x_{k+1}$  for all  $k \in \llbracket 1, N-1 \rrbracket$ , that sends all the other vectors of  $\mathbf{B}$  to 0, and that sends all the vectors of  $\mathbb{F}_{d-1}[t]x \oplus F'$  to 0.

Then, we define inductively  $(y_1, \dots, y_M)$  by  $y_1 := x_1$  and, for all  $k \in \llbracket 1, M-1 \rrbracket$ ,  $y_{k+1} := (u-f)(y_k)$ : this makes sense because one proves by induction that, for each  $k \in \llbracket 1, M-1 \rrbracket$ , the vector  $y_k$  equals the  $k$ -th vector of  $\mathbf{B}$  modulo the sum of  $\mathbb{F}_{m+k-2}[t]x \oplus F'$  with the span of the first  $k-1$  vectors of  $\mathbf{B}$ . It follows that  $y_M$  equal  $u^{n_N-1}(x_N)$  modulo  $\mathbb{F}_{d-2}[t]x \oplus F' \oplus G_1$ . Setting

$$G_2 := G_1 \oplus \mathbb{F}y_M,$$

we note that

$$W = F \oplus G_2$$

and that  $u(y_M) = z + z'$  for some  $(z, z') \in (\mathbb{F}_{d-1}[t]x \oplus F') \times G_2$ .

We finally extend  $f$  into a linear map on  $G_2 \oplus \mathbb{F}_{d-1}[t]x \oplus F'$  by setting

$$f(y_M) := ay_M + z - x,$$

so that  $(u - f)(y_M) = x$  modulo  $G_2$ . Now, set

$$V' := \mathbb{F}[t]t^d x \oplus H.$$

Applying the sewing lemma to the endomorphism of  $V'$  induced by  $u$  and to the vector  $t^d x$ , we recover an endomorphism  $g$  of  $V'$  such that  $g^2 = ag$  and the sequence  $((u - g)^k(t^d x))_{k \in \mathbb{N}}$  spans  $V'$ .

Finally, noting that  $V = \mathbb{F}_{d-1}[t]x \oplus V' \oplus G_2 \oplus F'$ , we consider the unique endomorphism  $v$  of  $V$  whose restriction to  $V'$  is  $g$  and whose restriction to  $\mathbb{F}_{d-1}[t]x \oplus F' \oplus G_2$  is  $f$ .

We claim that  $v^2 = av$  and that  $u - v$  is elementary. Let us check first the equality on each subspace  $V'$ ,  $\mathbb{F}_{d-1}[t]x \oplus F'$  and  $G_2$ . First of all, both  $v^2$  and  $av$  vanish everywhere on  $\mathbb{F}_{d-1}[t]x \oplus F'$ . Next, by the very definition of  $g$ , we know that  $v^2$  and  $av$  coincide on  $V'$ . Finally, it is easily checked that  $v^2(y) = av(y)$  for every vector  $y$  in  $\mathbf{B}$  by using the fact that  $n_2, \dots, n_N$  are all greater than 1; on the other hand, since  $z - x$  belongs to  $\mathbb{F}_{d-1}[t]x \oplus F'$ , we have  $f(z - x) = 0$  and it follows that  $v((v - a \text{id})(y_M)) = 0$ , i.e.  $v^2(y_M) = av(y_M)$ .

Obviously, the module  $(F')^{u-v} = (F')^u$  is free. In order to conclude, we shall simply check that  $\mathbb{F}[t]x \oplus G_2 \oplus H$  is stabilized by  $u - v$  and that the module  $(\mathbb{F}[t]x \oplus G_2 \oplus H)^{u-v}$  is free with generator  $y_1$ . First of all, we have  $(u - v)^i(y_1) = y_{i+1}$  for all  $i \in \llbracket 1, M - 1 \rrbracket$ , and then  $(u - v)^M(y_1) = x$  modulo  $\text{span}((u - v)^i(y_1))_{0 \leq i < M}$ . Then, as  $f$  vanishes everywhere on  $\mathbb{F}_{d-1}[t]x$ , the definitions of  $d$  and  $v$  show, by induction, that  $(u - v)^k(y_1) = u^{k-M}(x)$  modulo  $\text{span}((u - v)^i(y_1))_{0 \leq i < k}$  for all  $k \in \llbracket M, M + d - 1 \rrbracket$ . Moreover, the choice of  $g$  shows that

$$V' = \text{span}((u - v)^l(u^d(x)))_{l \in \mathbb{N}} \subset \text{span}((u - v)^l(y_1))_{l \in \mathbb{N}} \subset \mathbb{F}[t]x \oplus H \oplus G_2.$$

It follows that  $((u - v)^i(y_1))_{i \in \mathbb{N}}$  generates the infinite-dimensional vector space  $\mathbb{F}[t]x \oplus H \oplus G_2$ , which yields the claimed result. Therefore,  $V = F' \oplus (\mathbb{F}[t]x \oplus H \oplus G_2)$  is a free  $\mathbb{F}[t]$ -module for the structure induced by  $u - v$ , or in other words  $u - v$  is elementary.

Finally, assume that  $W = F$ . Then, we simply split  $V = (\mathbb{F}[t]x \oplus H) \oplus F'$ , and we apply the sewing lemma to the endomorphism of  $\mathbb{F}[t]x \oplus H$  induced by  $u$ , which yields an endomorphism  $w$  of  $\mathbb{F}[t]x \oplus H$  such that  $w^2 = aw$  and the

module  $(\mathbb{F}[t]x \oplus H)^{u-w}$  is free. We extend  $w$  into an endomorphism  $v$  of  $V$  that maps every vector of  $F'$  to 0, and we obtain that  $v^2 = av$  and that  $V^{u-v}$  is free.  $\square$

Now, the proof of Theorem 11 is complete over all vector spaces of countable dimension. Hence, Theorem 2 is finally established in all situations.

## 9 Special decompositions

In this last section, we complete the proofs of Theorems 5, 8 and 9.

### 9.1 Sums of three square-zero endomorphisms

Let  $u$  be an endomorphism of an infinite-dimensional vector space  $V$ . Let us first apply Theorems 2, 3 and 4 to  $p_1 = p_2 = p_3 = t^2$ . Here,  $\text{tr } p_1 + \text{tr } p_2 + \text{tr } p_3 = 0$ , and a scalar is a  $(p_1, p_2, p_3)$ -sum if and only if it equals 0. Hence:

- If  $u$  has no dominant eigenvalue then it is the sum of three square-zero endomorphisms, by Theorem 2.
- If  $u$  has a dominant eigenvalue  $\lambda$ , it is the sum of three square-zero endomorphisms only if  $\lambda = 0$  or  $\mathbb{F}$  has characteristic 2, according to Theorem 3.
- If  $u$  has a dominant eigenvalue  $\lambda$  such that  $u - \lambda \text{id}_V$  has infinite rank, and either  $\lambda = 0$  or  $\mathbb{F}$  has characteristic 2, then Theorem 4 yields that  $u$  is the sum of three square-zero endomorphisms.

It only remains to tackle the case when  $u$  splits as  $u = \lambda \text{id}_V + w$  for some finite-rank endomorphism  $w$  of  $V$ , and either  $\mathbb{F}$  has characteristic 2 or  $\lambda = 0$ .

Assume that  $u$  is the sum of three square-zero endomorphisms. Let  $A$  be a square matrix in the class  $[w]$  (see Section 4), with size  $n \times n$ . By Theorem 14, there is a non-negative integer  $q$  such that  $(A + \lambda I_n) \oplus \lambda I_q$  is the sum of three square-zero matrices. Hence, its trace equals zero, leading to  $\text{tr } w + (n+q)\lambda = 0$ . If  $\mathbb{F}$  has characteristic 2, this yields  $\text{tr } w \in \{0, \lambda\}$ , otherwise we know that  $\lambda = 0$  and hence  $\text{tr } w = 0$ .

Conversely, by Corollaries 1.5 and 1.6 of [3] we have the following results:

- Every finite-rank endomorphism  $w$  of  $V$  with trace zero is the sum of three square-zero endomorphisms of  $V$ .

- If  $\mathbb{F}$  has characteristic 2, then, for all  $\lambda \in \mathbb{F}$  and every finite-rank endomorphism  $w$  of  $V$  with trace in  $\{0, \lambda\}$ , the endomorphism  $\lambda \text{id}_V + w$  is the sum of three square-zero endomorphisms of  $V$ .

This completes the proof of Theorem 5.

## 9.2 Sums of three idempotents over a field with characteristic 2

Here, we assume that the underlying field  $\mathbb{F}$  has characteristic 2. We put  $p_1 = p_2 = p_3 = t^2 - t$ . Since  $\mathbb{F}$  has characteristic 2 the equation  $2\lambda = \text{tr } p_1 + \text{tr } p_2 + \text{tr } p_3$  has no solution in  $\mathbb{F}$ . Moreover, a scalar is a  $(p_1, p_2, p_3)$ -sum if and only if it belongs to  $\{0_{\mathbb{F}}, 1_{\mathbb{F}}\}$ . Hence:

- If  $u$  has no dominant eigenvalue then it is the sum of three idempotent endomorphisms, by Theorem 2.
- If  $u$  has a dominant eigenvalue  $\lambda$ , then it is the sum of three idempotent endomorphisms only if  $\lambda \in \{0_{\mathbb{F}}, 1_{\mathbb{F}}\}$ , according to Theorem 3.
- If  $u$  has a dominant eigenvalue  $\lambda \in \{0_{\mathbb{F}}, 1_{\mathbb{F}}\}$  such that  $u - \lambda \text{id}_V$  has infinite rank, then  $u$  is the sum of three idempotent endomorphisms, by Theorem 4.

It remains to consider the case when  $u = \lambda \text{id}_V + w$  for some finite-rank endomorphism  $w$  of  $V$  and some  $\lambda \in \{0_{\mathbb{F}}, 1_{\mathbb{F}}\}$ . Yet, every idempotent square matrix with entries in  $\mathbb{F}$  has its trace in  $\{0_{\mathbb{F}}, 1_{\mathbb{F}}\}$ . With the same line of reasoning as in Section 9.1, we obtain that if  $u$  is the sum of three idempotent endomorphisms of  $V$  then  $m\lambda + \text{tr}(w) \in \{0_{\mathbb{F}}, 1_{\mathbb{F}}\}$  for some non-negative integer  $m$ , whence  $\text{tr}(w) \in \{0_{\mathbb{F}}, 1_{\mathbb{F}}\}$ .

Conversely, by Corollary 1.7 from [3], for every  $\lambda \in \{0_{\mathbb{F}}, 1_{\mathbb{F}}\}$  and every finite-rank endomorphism  $w$  of  $V$  such that  $\text{tr } w \in \{0_{\mathbb{F}}, 1_{\mathbb{F}}\}$ , the endomorphism  $\lambda \text{id}_V + w$  is the sum of three idempotent endomorphisms.

This completes the proof of Theorem 8.

## 9.3 Every endomorphism is a linear combination of three idempotents

Theorem 9 is already known in the finite-dimensional case: See [4]. Now, we complete the infinite-dimensional case.

Let  $u$  be an endomorphism of an infinite-dimensional vector space  $V$ . If  $u$  has no dominant eigenvalue, then  $u$  is the sum of three idempotent endomorphisms, by Theorem 2. Assume now that  $u$  has a dominant eigenvalue  $\lambda$  such that  $u - \lambda \operatorname{id}_V$  has infinite rank. We can split  $\lambda = a_1 + a_2 + a_3$  where each  $a_i$  is a scalar in  $\mathbb{F}$ . For all  $i \in \llbracket 1, 3 \rrbracket$ , we set  $p_i := t^2 - a_i t$  if  $a_i \neq 0$ , otherwise we set  $p_i := t^2 - t$ . Hence,  $\lambda$  is a  $(p_1, p_2, p_3)$ -sum, and by Theorem 4 we conclude that  $u$  is a  $(p_1, p_2, p_3)$ -sum, which yields that it is a linear combination of three idempotents.

Assume finally that  $u$  has a dominant eigenvalue  $\lambda$  for which  $u - \lambda \operatorname{id}_V$  has finite rank. Then, Corollary 1.8 from [3] yields that  $u$  is a linear combination of three idempotents.

This completes the proof of Theorem 9.

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